

Vertical martingales, stochastic calculus and harmonic sections

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Abstract

This work is about a new class of martingales: the vertical martingales. We construct the vertical martingale for smooth submersions and we develop a stochastic calculus for one. Furthermore, we give a stochastic characterization for harmonic sections.

Key words: vertical martingales; harmonic sections; stochastic analysis on manifolds

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1 Introduction

Let $\pi : E \rightarrow M$ be a Riemannian submersion with totally geodesic fibers property and denote by ∇^g the Levi-Civita connection on E . Since π is a submersion, it is possible to define the vertical spaces by $V_p E = \ker(\pi_{*p})$, $p \in E$. Also, we define the vertical connection ∇^v on E by vertical projection of the Levi-Civita connection ∇^g into $V_p E$. With this hypothesis, C.Wood defines, in [14], a harmonic section being a section σ of π such that

$$\tau_\sigma^v = \text{tr} \nabla^v \sigma_*^v = 0,$$

where σ_*^v is the vertical projection of σ_* into VE . In fact, Wood shows that this definition is consistent with a minimal solution of the vertical energy functional.

Our work has its idea based on harmonic sections. We explain it. One knows that there is a stochastic characterization for harmonic maps (see for example [3] or [4]). In a nutshell, if M is a Riemannian manifold, N a smooth manifold with a symmetric connection and $\phi : M \rightarrow N$ a smooth map, then ϕ is a harmonic map if and only if ϕ sends Brownian motions in M into martingales in N . However, harmonic sections ask the vertical connection on the target manifold. So, to construct a stochastic characterization for harmonic sections is necessary a new concept of the martingale: the vertical martingale.

The environment of our work is general. Let E, M be smooth manifolds such that there exists a submersion $\pi : E \rightarrow M$. We also endow E with a symmetric connection such that π has totally geodesic fibers property. The stochastic calculus in manifolds says that to define a martingale is necessary the concept

of the integral of Itô. In this way, before to define the vertical martingales we need to construct the vertical Itô integral. Based in the Rank theorem, we can define the vertical Itô integral using the Schwartz Theory. Furthermore, we define the vertical Stratonovich integral and to show a formula of changes between both.

A way to show the stochastic characterizations of harmonic maps is to use the geometric Itô formula (see for example [3]). In the same line, we construct a geometric Itô formula for the vertical Itô integral and the vertical Stratonovich integral. Both are useful. The geometric Itô formula is used to gives a stochastic characterization for the vertical harmonic maps. A smooth map $\phi : N \rightarrow E$ is called a vertical harmonic map if $\tau_\phi^v = \text{tr} \nabla^v \phi_*^v$ vanishes, where N is a Riemannian manifold.

The stochastic characterization of the vertical harmonic map gives directly that a section $\sigma : M \rightarrow E$ of π , where M is a Riemannian manifold, is harmonic section if and only if σ sends Brownian motions into vertical martingales.

As application we study the vertical martingales in the tangent space TM endowed with the complete lift connection or the Sasaky metric and, consequently, we conclude that every harmonic section with values in TM is the 0-section. Furthermore, we study the vertical martingales in the Riemannian principal fiber bundle.

2 Preliminaries

We begin by recalling some fundamental facts on Schwartz Theory and stochastic calculus on manifolds. We shall use freely concepts and notations from S. Kobayashi and N. Nomizu [8], L. Schwartz [13], P.A. Meyer [10] and M. Emery [4]. A quick survey in these concepts is described by P. Catuogno in [3].

Let M be a smooth manifold and $x \in M$. The second order tangent space to M at x , which is denoted by $\tau_x M$, is the vector space of all differential operators on M at x of order at most two without a constant term. Let (x_1, \dots, x_n) be a local system of coordinates. Every $L \in \tau_x M$ can be written in a unique way as

$$L = a_{ij} D_{ij} + a_i D_i,$$

where $a_{ij} = a_{ji}$, $D_i = \frac{\partial}{\partial x^i}$ and $D_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$ are differential operators at x (we shall use the convention of summing over repeated indices). The elements of $\tau_x M$ are called second order tangent vectors at x , the elements of the dual vector space $\tau_x^* M$ are called second order forms at x .

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ (respectively, $\tau^* M = \bigcup_{x \in M} \tau_x^* M$) is canonically endowed with a vector bundle structure over M , which is called the second order tangent fiber bundle (respectively, second order cotangent fiber bundle) of M .

Let M, N be smooth manifolds, $F : M \rightarrow N$ a smooth map and $L \in \tau_x M$. The differential of F , $F_*(x) : \tau_x M \rightarrow \tau_{F(x)} N$, is given by

$$F_*(x)L(f) = L_x(f \circ F),$$

where $f \in \mathcal{C}^\infty(N)$.

Let L be a second order vector field on M . The square operator of L , denoted by QL , is the symmetric tensor given by

$$QL(f, g) = \frac{1}{2}(L(fg) - fL(g) - gL(f)),$$

where $f, g \in C^\infty(M)$. Let $x \in M$. We consider $Q_x : \tau_x M \rightarrow T_x M \odot T_x M$ as the linear application defined by

$$Q_x(L = a_{ij}D_{ij} + a_i D_i) = a_{ij}D_i \odot D_j.$$

Push forward of the second order vectors by smooth maps is related to the so called Schwartz morphisms between second order tangent vector bundles.

Definition 2.1 *Let M and N be smooth manifolds, $x \in M$ and $y \in N$. A linear application $F : \tau_x M \rightarrow \tau_y N$ is called Schwartz morphism if*

1. $F(T_x M) \subset T_y N$;
2. for all $L \in \tau_x M$ we have $Q(FL) = (F \otimes F)(QL)$.

A linear application $F : \tau_x M \rightarrow \tau_y N$ is a Schwartz morphism if and only if there exists a smooth map $\phi : M \rightarrow N$ with $\phi(x) = y$ such that $F = \phi_{x*}$ (see for example Proposition 1 in [5]).

Let $(\Omega, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space which satisfies the usual conditions (see for instance [4]).

Definition 2.2 *Let M be a smooth manifold and X a stochastic process with values in M . We call X a semimartingale if, for all f smooth on M , $f(X)$ is a real semimartingale.*

L. Schwartz has noticed, in [13], that, if X is a continuous semimartingale in a smooth manifold M , the Itô's differentials dX_i and $d[X_i, X_j]$ (where (x_i) is a local coordinate system and X_i is the i th coordinate of X in this system) behave under a change of coordinates as the coefficients of a second order tangent vector. The (purely formal) stochastic differential

$$d^2 X_t = dX_t^i D_i + \frac{1}{2} d[X^i, X^j]_t D_{ij},$$

is a linear differential operator on M , at X_t , of order at most two, with no constant term. Therefore, the tangent object to X_t is formally one of second order. This fact is known as Schwartz principle.

From now on we assume that all semimartingales are continuous.

Let X be a semimartingale in M . Let $\Theta_{X_t} \in \tau_{X_t}^* M$ be an adapted stochastic second order form along X_t . Let (U, x^i) be a local coordinate system in M . With respect to this chart the second order form Θ can be written as

$\Theta_x = \Theta_i(x)d^2x^i + \Theta_{ij}(x)dx^i \cdot dx^j$ where Θ_i and $\Theta_{ij} = \Theta_{ji}$ are (\mathcal{C}^∞ say) functions in M . Then the integral of Θ along X is defined by

$$\int_0^t \Theta d^2X = \int_0^t \Theta_i(X_s)dX_s^i + \int_0^t \Theta_{ij}(X_s)d[X^i, X^j]_s. \quad (1)$$

Let b be a section of $T_0^2(M)$, which is defined along X . The quadratic integral of b along X is defined, locally, by

$$\int_0^t b(dX, dX) = \int_0^t b_{ij}(X_t)d[X^i, X^j]_t,$$

where $b(x) = b_{ij}(x)dx^i \otimes dx^j$ and b_{ij} are smooth functions. Here, we observe that both the integral $\int_0^t \Theta d^2X$ and $\int_0^t b(dX, dX)$ are well defined. To see these facts we refer the reader to [4].

Let M be a smooth manifold endowed with symmetric connection ∇^M . In [10], P. Meyer showed that for ∇^M there exists a section Γ^M in $Hom(\tau M, TM)$ such that $\Gamma^M|_{TM} = Id_{TM}$ and $\Gamma^M(AB) = \nabla_A^M B$, where $A, B \in TM$. We also say Γ^M of connection.

Let M be a smooth manifold endowed with a symmetric connection Γ^M . Let X be a semimartingale in M and θ be a 1-form along X . The Itô integral is defined by $\int_0^t \theta d^M X := \int_0^t \Gamma^{M*} \theta d^2 X_t$. Furthermore, X is said ∇^M -martingale if for every 1-form θ on M we have that $\int_0^t \theta d^M X$ is a real local martingale.

Definition 2.3 *Let M be a Riemannian manifold with metric g . A semimartingale B in M is called Brownian motion if $\int_0^t \theta d^g B_t$ is a real local martingale for all $\theta \in T^*M$, where Γ^g is the Levi-Civita connection, and for any section b in $T_0^2(M)$ we have*

$$\int_0^t b(dB, dB) = \int_0^t \text{tr } b_{B_s} ds. \quad (2)$$

3 Vertical integral of Itô and Stratonovich

Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let us denote the vertical distribution by $VE = \ker(\pi_*)$ and vertical projection by $\mathbf{v} : TE \rightarrow VE$. A vector field X on E is called vertical if $X_p \in V_p E$, $p \in E$. Analogous, a 1-form θ is called vertical form if $\theta(p) \in V_p^* E$.

Let $p \in E$, by Rank Theorem, there exist a coordinate $(x_1, \dots, x_m, v_1, \dots, v_k)$ of some neighborhood $U \ni p$ such that

$$\pi(x_1, \dots, x_m, v_1, \dots, v_k) = (x_1, \dots, x_m), \quad (3)$$

where (x_1, \dots, x_m) is a coordinate of a neighborhood $V \ni \pi(p)$. The possibility of choice these coordinates, is fundamental to construct the vertical martingale.

Before going further the construction of the vertical integrals we want to know about behavior of the vertical vectors fields and of the vertical forms. A

well known property of the submersions assures that for each p in E there exists a smooth local section σ of π where p is in the image of σ (see for instance Proposition 5.18 in [9]). In other words, the Rank theorem assures that exists a neighborhood V in M and a smooth local section $\sigma : V \rightarrow E$ such that, in coordinates (3), $\sigma(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$. Let $i : \sigma(U) \hookrightarrow E$ be the inclusion map and v^α , $\alpha = 1, \dots, k$, the coordinate functions given by (3). Then

$$\begin{aligned} v^\alpha|_{\sigma(U)} &= v^\alpha \circ i(\sigma(U)) \\ &= v^\alpha \circ i(\sigma(x_1, \dots, x_k)) \\ &= v^\alpha \circ i((x_1, \dots, x_k, 0, \dots, 0)) \\ &= v^\alpha(x_1, \dots, x_k, 0, \dots, 0) = 0. \end{aligned} \tag{4}$$

Suppose that X is a vertical vector field, so in coordinates (3) it is written as

$$X = a^\alpha \frac{\partial}{\partial v_\alpha}.$$

By a well known account, $a_\alpha = dv^\alpha(X)$. Thus, from (4) we see that a_α is constant over $\sigma(V)$. Consequently, X is constant over $\sigma(V)$.

Let θ be a vertical form. In coordinates (3), θ is written as

$$\theta = \theta^\alpha dv^\alpha.$$

Inducing θ in $\sigma(V)$ we have

$$i^*\theta = (\theta^\alpha \circ i)dv^\alpha \circ i.$$

This imply that $i^*\theta = 0$. Then this allows consider $(\theta^\alpha \circ i)$ as a constant function. So θ is constant over $\sigma(V)$. It is the key to prove the Itô formula for Stratonovich integral in the next section. Thus the constance of the vertical forms and vector field is a local behavior. In the case that Rank Theorem be global, for example $E = M \times N$, then the constance is global.

Taking the coordinates (3) we obtain in $\tau_p E$ the coordinate basis $\{\bar{D}_1(p), \dots, \bar{D}_m(p), D_1(p), \dots, D_k(p)\}$, where $\bar{D}_i = \partial/\partial x^i$, $i = 1, \dots, m$, and $D_\alpha = \partial/\partial v^\alpha$, $\alpha = 1, \dots, k$. It is clear that $\{D_1(p), \dots, D_k(p)\}$ are vertical vectors in $T_p E$ and it also spans the vertical space $V_p E$. Also, in coordinates (3), a second vector L in $\tau_p E$ is written as

$$L(p) = a_{\alpha\beta} D_{\alpha\beta}(p) + a_\alpha D_\alpha(p) + a_{ij} D_{ij}(p) + a_i D_i(p) + a_{\alpha j} D_{\alpha j}(p).$$

We denote by $\mathfrak{V}_p E$ the subspace spanned by $\{D_{\alpha\beta}, D_\alpha; \alpha, \beta = 1, \dots, k\}$. Here, we observe that $\mathfrak{V}_p M$ is not the $\ker \pi_*(p)$ because besides $\{D_{\alpha\beta}, D_\alpha; \alpha, \beta = 1, \dots, k\}$ being in $\ker \pi_*(p)$ also $\{D_{\alpha j}, \alpha = 1, \dots, k, j = 1, \dots, m\} \in \ker \pi_*(p)$. It is easily to see that $\mathfrak{V}_p E$ is the second order tangent vector field of the fiber $\pi^{-1}(x)$ at point p , where $\pi(p) = x$. We denote by $\mathfrak{V}E = \bigcup_{p \in E} \mathfrak{V}_p E$ and by $\mathbf{v} : \tau E \rightarrow \mathfrak{V}E$ the vertical projection.

Given a symmetric connection ∇^E on E , in each fiber $\pi^{-1}(x)$, $x \in M$, we can induce a connection ∇^x from the connection ∇^E . In our work, we suppose

that all fibers are totally geodesics. In this way, ∇^x is the vertical projection of ∇^E at $p \in \pi^{-1}(x)$, for vertical vector fields. For our purpose we generalize this concept in the following way: let U, V be vertical vector fields on E , we define the vertical connection on E by $\nabla_U^v V = \mathbf{v}\nabla_U^E V$. It is easily to see that $(\nabla_U^v V)(p) = (\nabla_U^x V)(p)$.

Now we want to see the vertical connection in the context of second order. As Γ^E is a linear homomorphism from τE into TE we can restrict Γ^E to $\mathfrak{V}E$. A little bit more, we take $\Gamma^v(L) = \mathbf{v}\Gamma^E(L)$, where $L \in \mathfrak{V}E$. The connection Γ^v is the object that is associated with ∇^v . In fact, if U, V are vector fields on E , then UV is a second vector field in $\mathfrak{V}E$. A simple account in coordinates gives $\Gamma^v(UV) = \nabla_U^v V$.

Our next step is to construct an integral of Itô for the vertical connections. Let X be a semimartingale in E . Adopting the coordinates (3) we obtain, by Schwartz principle,

$$d^2 X_t = dX_t^\alpha D_\alpha + \frac{1}{2} d[X^\alpha, X^\beta]_t D_{\alpha\beta} + dX_t^i D_i + \frac{1}{2} d[X^i, X^j]_t D_{ij} + \frac{1}{2} d[X^i, X^\beta]_t D_{i\beta}.$$

Since that $d^2 X_t \in \tau_{X_t} M$, we can project it into $\mathfrak{V}_{X_t}(E)$, that is,

$$\mathbf{v}(d^2 X_t) = dX_t^\alpha D_\alpha + \frac{1}{2} d[X^\alpha, X^\beta]_t D_{\alpha\beta}.$$

Let Θ_{X_t} be an adapted stochastic second order form along X_t such that $\Theta_{X_t} \in \mathfrak{V}_{X_t}^* M$. We take the coordinates (3) in E . With respect to these coordinates the second order form Θ can be written as $\Theta_p = \Theta_\alpha d^2 x^\alpha + \Theta_{\alpha\beta} dx^\alpha \cdot dx^\beta$ where Θ_α and $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$ are (\mathcal{C}^∞ say) functions in E . From definition of the integral (1) we can see that

$$\int_0^t \Theta d^2 X_t = \int_0^t \Theta_\alpha(X_s) dX_s^\alpha + \int_0^t \Theta_{\beta\gamma}(X_s) d[X^\beta, X^\gamma]_s = \int_0^t \Theta \mathbf{v} d^2 X_t. \quad (5)$$

We observed that the integral $\int_0^t \Theta \mathbf{v} d^2 X_t$ is well posed. In the sense that it is based in the $\int_0^t \Theta d^2 X_t$, which has a good definition (see for example Theorem 2.10 in [5] or section 4. in [10]).

Now, since $\mathbf{v} d^2 X_t \in \mathfrak{V}_{X_t}(E)$, there is sense the $\Gamma^v(\mathbf{v}(d^2 X_t))$. A little bit more, if θ is a vertical form in E , taking the coordinates (3) we can write $\theta(p) = \theta_\alpha(p) dv^\alpha$ and, consequently, $\Gamma^{v*}\theta(p) = \theta^\alpha(p)(d^2 v^\alpha + \Gamma_{\beta\gamma}^\alpha(p) dv^\beta \cdot dv^\gamma)$. It follows that every tools to define a Itô integral for vertical connections are well posed.

Definition 3.1 *Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let ∇^E be a symmetric connection, X a semimartingale on E and θ a vertical form on E . Assume that π has totally geodesic fibers property. We define the vertical Itô integral of θ along X as $\int \theta d^v X_t = \int \Gamma^{v*}\theta(\mathbf{v} d^2 X_t)$. Let (U, x_1, \dots, x_n) be a chart such that (3) is true. Then, locally, the vertical Itô integral of θ along X is given by*

$$\int \theta d^v X_t = \int_0^t \theta_\alpha(X_s) dX_s^\alpha + \int_0^t \Gamma_{\beta\gamma}^\alpha \theta_\alpha(X_s) d[X^\alpha, X^\beta]_s.$$

One can observe that the vertical Itô integral is well posed because the right side of the above equality is an integral as (5).

From the Definition above we can define our main object: the vertical martingales.

Definition 3.2 *A semimartingale in E is called vertical martingale if $\int \theta d^v X_t$ is a real local martingale for every vertical form on E .*

Our next step is to define a vertical Stratonovich integral. Let U be a chart in E with coordinates (3). Let θ be a vertical form on E , so in coordinates we have $\theta = \theta_\alpha dv^\alpha$. It is well known to define the Stratonovich integral we need to yields a second order form from one form θ . For such, we use the operator $d : T^*E \rightarrow \tau^*E$ which is, locally, given by

$$d\theta(p) = \theta_\alpha(p)d^2v^\alpha + D_\alpha\theta_\beta(p)dv^\alpha \cdot dv^\beta + D_i\theta_\alpha(p)dx^i \cdot dv^\alpha.$$

Now, applying the projection \mathbf{v} at $d\theta$ we obtain

$$\mathbf{v}d\theta(p) = \theta_\alpha(p)d^2v^\alpha + D_\alpha\theta_\beta(p)D_{\alpha\beta}.$$

In this way $\mathbf{v}d\theta(p) \in \mathcal{V}^*E$. Taking a semimartingale X in E there is sense in

$$\mathbf{v}d\theta(p)(\mathbf{v}d^2X_t) = \theta_\alpha(X_t)dX^\alpha + \frac{1}{2}D_\alpha\theta_\beta(X_t)d[X^\alpha, X^\beta].$$

Therefore for (5) we have a good definition for the vertical Stratonovich integral.

Definition 3.3 *Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let X be a semimartingale on E and θ a vertical form on E along X . We define the vertical Stratonovich integral of θ along X as $\int \theta \delta^v X_t = \int \mathbf{v}d\theta(\mathbf{v}d^2X_t)$. Let (U, x_1, \dots, x_n) be a chart such that (3) is true. Then, locally, the vertical Stratonovich integral of θ along X is given by*

$$\int_0^t \theta \delta^v X_t = \int_0^t \theta_\alpha(X_s)dX_s^\alpha + \int_0^t D_\alpha\theta_\beta(X_s)d[X^\alpha, X^\beta]_s.$$

From definition of the vertical Itô and Stratonovich integrals we can show a change formula between they. In fact, a simple computation in coordinates gives

Proposition 3.1 *For a vertical form θ on E and a semimartingale X on E we have*

$$\int \theta \delta^v X_t = \int \theta d^v X_t - \frac{1}{2} \int \nabla^v \theta(dX, dX).$$

4 The geometric Itô formulas

At begin of the previous section we see that, locally, vertical vector fields and vertical forms have a constance property. An immediate interest in this fact is due the next build: the Itô formula for the vertical Stratonovich integral. Let θ be a vertical form in E , so, in coordinates (3), we have that $\theta = \theta^\alpha dv^\alpha$. By constance property of θ , θ^α is constant along coordinates (x^1, \dots, x^m) . In consequence, $\partial\theta^\alpha/\partial x_i \equiv 0$. This is useful in the following

Lemma 4.1 *Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$, $\phi : N \rightarrow E$ a smooth map and θ a vertical form on E . Then*

$$\phi^* \mathbf{v} d\theta = d(\phi^* \theta).$$

Proof: We first adopt the coordinates (3). It is direct that $(\phi^* \theta) = (\theta^\alpha \circ \phi) d(v^\alpha \circ \phi)$. Applying the operator d at $(\phi^* \theta)$ we compute

$$\begin{aligned} d(\phi^* \theta) &= d((\theta^\alpha \circ \phi) d(v^\alpha \circ \phi)) \\ &= d(\theta^\alpha \circ \phi) d(v^\alpha \circ \phi) + (\theta^\alpha \circ \phi) d^2(v^\alpha \circ \phi) \\ &= \phi^* d\theta^\alpha \phi^* dv^\alpha + (\theta^\alpha \circ \phi) \phi^* d^2 v^\alpha \\ &= \phi^* (d\theta^\alpha dv^\alpha + \theta^\alpha d^2 v^\alpha) \end{aligned}$$

Now the differential of θ^α is given by

$$d\theta^\alpha = \frac{\partial \theta^\alpha}{\partial x_i} dx^i + \frac{\partial \theta^\alpha}{\partial v_\beta} dv^\beta.$$

By constance property of θ ,

$$d\theta^\alpha = \frac{\partial \theta^\alpha}{\partial v_\beta} dv^\beta.$$

We thus obtain

$$d(\phi^* \theta) = \phi^* \left(\frac{\partial \theta^\alpha}{\partial v^\beta} dv^\beta dv^\alpha + \theta^\alpha d^2 v^\alpha \right).$$

Since the right side, in coordinates, is $\phi^*(\mathbf{v} d\theta)$, we conclude that

$$d(\phi^* \theta) = \phi^*(\mathbf{v} d\theta).$$

Lemma above has an analogous when the operator d acts in any first order form (see Proposition 1.18 in [5]). Furthermore, the result of Lemma above for any first order form is responsible to the Itô formula for Stratonovich integral (see for example [5] or [10]). Following the same idea, Lemma above was constructed specifically to show

Theorem 4.2 *Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$, $\phi : N \rightarrow E$ a smooth map and θ a vertical form on E . Then*

$$\int \theta \delta^v \phi(X) = \int (\phi^* \theta) \delta X. \quad (6)$$

Proof: Let X be a semimartingale in N and θ a vertical form on E . By definition of the vertical Stratonovich integral,

$$\int \theta \delta^v \phi(X) = \int \mathbf{v} d\theta d^2 \phi(X) = \int \phi^* \mathbf{v} d\theta d^2 X = \int d(\phi^* \theta) d^2 X = \int (\phi^* \theta) \delta X,$$

where we used Lemma 4.1 in the third equality.

Beyond the Itô formula for Stratonovich integral the stochastic calculus in manifolds has another formula for transformation between manifolds: the geometric Itô formula (see for example [3]). Our next purpose is to construct the geometric Itô formula for the vertical Itô integral. For this construct we follow [3]. We begin introducing the second fundamental form, tension field and vertical harmonic map.

Definition 4.1 Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$ and $\phi : N \rightarrow E$ a smooth map. Suppose that E is equipped with a symmetric connection ∇^E such that π has totally geodesics fibers property and N is equipped with symmetric connection ∇^N . Furthermore, denote the vertical connection by Γ^v . The section α_ϕ^v of $\mathfrak{V}^* E \otimes \phi^* V E$ is given by

$$\alpha_\phi^v = \Gamma^v \mathbf{v} \phi_* - \mathbf{v} \phi_* \Gamma^N. \quad (7)$$

The vertical second fundamental form of ϕ , β_ϕ^v , is the unique section of $(TM \odot TM)^* \otimes \phi^* V E$ such that $\alpha_\phi^v = \beta_\phi^v \circ Q$. The tension field of β_ϕ^v is

$$\tau_\phi^v = \text{tr} \beta_\phi^v.$$

We call ϕ a vertical harmonic map if $\tau_\phi^v = 0$.

The following linear algebra lemma shows that β_F^v is well defined.

Lemma 4.3 Let α_ϕ^v be a section of $\mathfrak{V}^* E \otimes \phi^* V E$ defined by (7). Then there exists an unique section β_ϕ^v of $(TM \odot TM)^* \otimes \phi^* V E$ such that $\alpha_\phi^v = \beta_\phi^v \circ Q$.

Proof: Since $\text{Ker } Q = TM \subset \text{Ker } \alpha_\phi^v$, the lemma follows from the first isomorphism theorem (see [12] pp 67).

The following lemma is necessary in the main Theorem of this section.

Lemma 4.4 Under assumptions in Definition 4.1, for each vertical form θ on E ,

$$\int \alpha_\sigma^{v*} \theta \, d_2 X = \frac{1}{2} \int \beta_\sigma^{v*} \theta (dX, dX).$$

Proof: By definition of β_ϕ^v , for each vertical form θ we have

$$\frac{1}{2} \int \beta_\sigma^{v*} \theta (dX, dX) = \int Q^* \beta_\sigma^{v*} \theta \, d^2 X = \int (\beta_\sigma^v \circ Q)^* \theta \, d^2 X = \int \alpha_\sigma^{v*} \theta \, d^2 X.$$

The first equality follows from Proposition 6.31 in [4].

Let us make a clear observation. Lemma 4.3 assures an existence β_ϕ^v , in the other hand, Lemma 4.4 shows as related the integral of α_ϕ^v and β_ϕ^v . Both results are vital for the construction of the following geometric Itô formula.

Theorem 4.5 *Let E, M, N be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$ and $\phi : N \rightarrow E$ a smooth map. Suppose that E is equipped with a symmetric connection ∇^E such that π has totally geodesics fibers property and N is equipped with a symmetric connection ∇^N . Furthermore, denote the vertical connection by ∇^v . If X is an N -valued semimartingale and θ is a vertical form on E , then*

$$\int \theta d^v \phi(X) = \int \phi^* \theta d^N X + \frac{1}{2} \int \beta_\phi^{v*} \theta(dX, dX).$$

Proof: We calculate

$$\begin{aligned} \int \theta d^v \phi(X) &= \int (\Gamma^{v*} \theta) \mathbf{v} d^2 \phi(X) \\ &= \int (\Gamma^{v*} \theta) \mathbf{v} \phi_* d^2(X) = \int \mathbf{v} \phi^* (\Gamma^{v*} \theta) d^2 X \\ &= \int \mathbf{v} \phi^* (\Gamma^{v*} \theta) d^2 X + \int \Gamma^{N*} (\phi^* \theta) d^2 X - \int \Gamma^{N*} (\phi^* \theta) d^2 X \\ &= \int \Gamma^{N*} (\phi^* \theta) d^2 X + \int (\mathbf{v} \phi^* (\Gamma^{v*} \theta) - \Gamma^{N*} (\phi^* \theta)) d^2 X \\ &= \int \phi^* \theta d^N X + \int \alpha_\phi^{v*} \theta d_2 X \\ &= \int \phi^* \theta d^N X + \frac{1}{2} \int \beta_\phi^{v*} \theta(dX, dX), \end{aligned}$$

where we use Lemma 4.4 in the last equality.

Corollary 4.6 *Under assumptions of Theorem 4.5, furthermore, (N, g) is a Riemannian and ∇^g is the Levi-Civita connection on N . If B is a g -Brownian motion in N and θ be a vertical form on E , then*

$$\int \theta d^v \phi(B) = \int \phi^* \theta d^g B + \frac{1}{2} \int \tau_\phi^{v*} \theta(B_t) dt. \quad (8)$$

A link between Stochastic Analysis and Differential Geometry is a well known stochastic characterization of harmonic maps, see for example [4] or [5]. One can observe that until here we construct analogous tools to show a stochastic characterization to vertical harmonic maps.

Proposition 4.7 *Let E, M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Suppose that E is equipped with a symmetric connection ∇^E such that π has totally geodesics fibers property and (N, g) is a Riemannian manifold. Denote the vertical connection by ∇^v . A smooth map $\phi : N \rightarrow E$ is vertical harmonic map if and only if ϕ sends g -Brownian motions in vertical martingales $\phi(B)$.*

Proof: Let B be a g -Brownian motion in N and θ a vertical form on E . By formula (8),

$$\int_0^t \theta d^v \phi(B_s) = \int_0^t \phi^* \theta d^g B_s + \frac{1}{2} \int_0^t \tau_\phi^{v*} \theta(B_s) ds.$$

We observe that $\int \phi^* \theta d^g B_s$ is a real local martingale. Since B and θ are arbitrary, the Doob-Meyer decomposition assures that $\int_0^t \theta d^v \phi(B_s)$ is a real local martingale if and only if τ_ϕ^{v*} vanishes. From the definitions of vertical martingales and vertical harmonic maps we conclude the proof. \square

5 Harmonic section

In this section, we work with object that motivate this study: harmonic section. Before going further, we give the environment for the study of the harmonic sections.

Let E be a differential manifold and (M, g) a Riemannian manifold such that there is a smooth submersion $\pi : E \rightarrow M$. Let ∇^E be a symmetric connection on E such that π has totally geodesics fibers property and denote by ∇^g the Levi-Civita connection on M . Let $VE = \ker(\pi_*)$ be the vertical distribution and HE a smooth distribution in TE such that $TE = VE \oplus HE$. Let $\mathbf{v} : TE \rightarrow VE$ and $\mathbf{h} : TE \rightarrow HE$ be the vertical and horizontal projectors, respectively. Let $H_p = (\pi_p|_{H_p E})^{-1} : T_x M \rightarrow H_p E$ be the horizontal lift, where $\pi(p) = x$. We observe that H_p is an isomorphism for each $p \in E$. The submersion $\pi : E \rightarrow M$ is called affine submersion with horizontal distribution if $\mathbf{h}\nabla_{H(X)}^E H(Y) = H(\nabla_X^M Y)$ (see [1] for more details). A Riemmanian submersion is a classical example of affine submersion with horizontal distribution.

In this section, unless otherwise stated, we assume that $\pi : E \rightarrow M$ is an affine submersion with horizontal distribution.

Next we extend the definition given by C. M. Wood [14] for harmonic sections.

Definition 5.1 *A section σ of π is called a harmonic section if $\tau_\sigma^v = 0$.*

An immediate consequence of Proposition 4.7 is the characterization of the harmonic sections in the following stochastic context.

Theorem 5.1 *Let E be a differential manifold and (M, g) a Riemannian manifold such that there is a smooth submersion $\pi : E \rightarrow M$. Assume that E is equipped with a symmetric connection ∇^E such that π has totally geodesics fibers property. Then a section σ of π is harmonic section if and only if, for every g -Brownian motion B in M , $\sigma(B)$ is a vertical martingale in E .*

6 Applications

Tangent Bundle with Complete lift

Let (M, g) be a Riemannian manifold and TM its tangent bundle. To study the vertical martingales in TM we need to introduce a connection on it. Denoting by ∇^g the Levi-Civita connection we prolong ∇^g to the complete lift ∇^c on TM (see [15] for the definition of ∇^c). In a nutshell, if X, Y are vector fields on M , then ∇^c satisfies the following equations:

$$\begin{aligned} \nabla_{X^v}^c Y^v &= 0 \\ \nabla_{X^v}^c Y^h &= 0 \\ \nabla_{X^h}^c Y^v &= (\nabla_X Y)^v \\ \nabla_{X^h}^c Y^h &= (\nabla_X Y)^h + \gamma(R(-, X)Y, \end{aligned} \tag{9}$$

where $R(-, X)Y$ denotes a tensor field W of type $(1,1)$ on M such that $W(Z) = R(Z, X)Y$ for any $Z \in T^{(0,1)}(M)$, and γ is a lift of tensors, which is defined at page 12 in [15]. Furthermore, X^v, Y^v and X^h, Y^h are the vertical and horizontal lift of the X, Y , respectively.

Let $(\pi^{-1}(U), x, y)$ be a local coordinate system on TM and $(T\pi^{-1}(U), x, y, dx, dy)$ the induced local coordinate system on TTM . The vertical space is $V_{(a,u)}TM = \ker((\pi_*)_{(u,a)})$ and, in coordinates, a simple computation shows that $V_{(a,u)}TM$ is characterized by $dx = 0$ for $i = 1, \dots, n$. Also, the coordinates vector $\{D_\alpha = \frac{\partial}{\partial y_\alpha}; \alpha = 1, \dots, n\}$ at (a, u) span the vertical space $V_{(a,u)}TM$. Here, being TM a fiber bundle, the coordinates (3) always exist.

A direct account shows that the vertical connection ∇^v vanishes and that $\pi : TM \rightarrow M$ is a submersion with totally geodesic fibers property.

Proposition 6.1 *Let (M, g) be a Riemannian manifold and TM its tangent bundle equipped with the complete lift ∇^c . A semimartingale X_t in TM is vertical martingale if and only if, for each local coordinate system (TU, x, y) , $y(X_t)$ is a local martingale.*

Proof: We first suppose that X is a semimartingale in TM . A differential topology argument extend the map y for all TM . Thus the change formula and the Itô formula for Stratonovich integral gives

$$\begin{aligned} \int \theta d^{\mathbb{R}^n} y(X_t) &= \int \theta \delta y(X_t) - \frac{1}{2} \int \nabla^{\mathbb{R}^n} \theta(dy(X), dy(X)) \\ &= \int y^* \theta \delta X_t - \frac{1}{2} \int \nabla^{\mathbb{R}^n} \theta(dy(X), dy(X)). \end{aligned}$$

Being θ a 1-form in \mathbb{R}^n , $y^* \theta$ is a vertical form on TM . Now, due to local constance property of $y^* \theta$ we obtain

$$\int \theta d^{\mathbb{R}^n} y(X_t) = \int y^* \theta \delta^v X_t - \frac{1}{2} \int \nabla^{\mathbb{R}^n} \theta(dy(X), dy(X)).$$

Applying the change formula, Proposition 3.1, follows that

$$\int \theta d^{\mathbb{R}^n} y(X_t) = \int y^* \theta d^v X_t + \frac{1}{2} \int \nabla^{v*} y^* \theta(dX, dX) - \frac{1}{2} \int \nabla^{\mathbb{R}^n} \theta(dy(X), dy(X)).$$

Because $\nabla^{\mathbb{R}^n}$ and ∇^v are flat connection we conclude, with a direct account, that

$$\int \theta d^{\mathbb{R}^n} y(X_t) = \int y^* \theta d^v X_t.$$

Thus this shows the Proposition.

Proposition 6.2 *Let M be a complete Riemannian manifold which is compact or has nonnegative Ricci curvature and TM its tangent bundle equipped with the complete lift ∇^c . A section σ is harmonic section if and only if σ is the 0-section.*

Proof: Let σ be a section of π . If σ is the 0-section, then σ is a harmonic map. Conversely, suppose that σ is a harmonic map. From Theorem 5.1 we see that $\sigma(B)$ is a vertical martingale. Taking a local coordinate system (TU, x, y) we obtain, from Proposition 6.1, that $y \circ \sigma(B_t)$ is a local martingale in \mathbb{R}^n . Because M is a complete Riemannian manifold which is compact or has nonnegative Ricci curvature, Theorem 6 in [7] assures that $y \circ \sigma$ is constant. We conclude that σ is the 0-section in (TU, x, y) because if it would not be the 0-section we translated it to 0-section. As σ is the 0-section for any local coordinate system we conclude that σ is globally the 0-section.

Tangent bundle with Sasaki metric

Let M be a Riemannian manifold and TM the tangent bundle equipped with the Sasaki metric g_s . See for example [6] for a complete study about Sasaki metric. Thus π_E is a Riemannian submersion with totally geodesic fibers property. From definition of the Sasaki metric we see that the vertical connection ∇^v vanishes. Thus, analogously to above application we obtain

Proposition 6.3 *Let M be a Riemannian manifold and TM its tangent bundle equipped with the Sasaki metric g_s . A semimartingale X_t in TM is vertical martingale if and only if, for each local coordinate system (TU, x, y) , $y(X_t)$ is a local martingale.*

Proposition 6.4 *Let M be a complete Riemannian manifold which is compact or has nonnegative Ricci curvature and TM its tangent bundle equipped with the Sasaki metric g_s . A section σ is harmonic section if and only if σ is the 0-section.*

Principal Riemannian fiber bundle

M. Arnaudon and S. Paycha, in [2], shows that semimartingales in a principal fiber bundle $P(M, G)$ with G -invariant Riemannian metric k can be decomposed into G - and M -valued semimartingales. More precisely, a semimartingale X with values in $P(M, G)$ splits in a unique way into a horizontal semimartingale X^h and a semimartingale V with values in G such that

$$X = X^h \cdot V.$$

Moreover, V is the stochastic exponential

$$V = \epsilon\left(\int \omega \delta X\right)$$

and X^h is the solution of the Itô equation

$$d^{\nabla^k} X^h = H_{\tilde{X}}^k d^{\nabla}(\pi \circ X).$$

Now we induce from the metric k , at each fiber $\pi^{-1}(x)$, $x \in M$, a metric k_x such that π has totally geodesic fibers. Also we will induce a metric h in

G such that $p : G \rightarrow \pi^{-1}(x) \subset P$ is an isometric map, for each $p \in P$. Let us denote by ∇^k, ∇^x and ∇^h the Levi-civita connections on $P, \pi^{-1}(x)$ and G , respectively. Observing that ∇^k induces a connection ∇^x at each fiber and the vertical connection ∇^v coincides with latter in the fibers we conclude that p is affine maps according ∇^h and ∇^v .

Lemma 6.5 *If θ is a vertical form on $P(M, G)$ and if A is a vector field on $P(M, G)$, then at each $p \in P$*

$$\nabla^{v*}\theta(A, A)_p = \nabla^{G*}\sigma(t_0)^*\theta(\xi, \xi)_{(\sigma(t_0)^{-1}p)},$$

where $A = A^h \cdot \xi$ with A^h is the horizontal lift of the $\pi_*(A)$ and ξ a vector field on G , furthermore, σ is the flow of A^h such that $A^h(\sigma(t)) = d\sigma/dt$ and $\sigma(t_0) = p$.

Proof: Let A be a vector fields. It is possible to write A as $A = A^h \cdot \xi$, where ξ is a vector field on G and A^h is the horizontal lift of $\pi_*(A)$. At each $p \in P$ we compute

$$\begin{aligned} \nabla^{v*}\theta(A, A)_p &= A_p\theta(A)_p - \theta_p(\nabla_A^v A(p)) \\ &= (A^h \cdot \xi)_p\theta(A^h \cdot \xi)_p - \theta_p(\nabla_{A^h \cdot \xi}^v A^h \cdot \xi(p)) \\ &= (\sigma(t_0)_*\xi)_p\theta(\sigma(t_0)_*\xi)_p - \theta_p(\nabla_{\sigma(t_0)_*\xi}^v \sigma(t_0)_*\xi(p)) \\ &= \xi_{(\sigma(t_0)^{-1}p)}A^{h*}\theta(\xi)_{(\sigma(t_0)^{-1}p)} - \theta(p)(A^h(\nabla_\xi^G \xi)_{(\sigma(t_0)^{-1}p)}) \\ &= \nabla^G A^{h*}\theta(\xi, \xi)_{(\sigma(t_0)^{-1}p)}, \end{aligned}$$

where σ is the flow of A^h such that $A^h(\sigma(t)) = d\sigma/dt$ and $\sigma(t_0) = p$.

Proposition 6.6 *Let X be a semimartingale in $P(M, G)$. X is a vertical martingale if and only if V is a ∇^G -martingale.*

Proof: Let X_t be a semimartingale in P such that $X = X^h \cdot V$ as was explained above. Take a vertical form θ on E . By Proposition 3.1,

$$\int \theta d^v X = \int \theta \delta^v(X^h \cdot V) - \frac{1}{2} \int \nabla^{v*}\theta(dX, dX).$$

A simple account yields

$$\int \theta \delta^v(X^h \cdot V) = \int R_V^* \theta \delta^v X^h + \int X^{h*} \theta \delta V = \int X^{h*} \theta \delta V,$$

where we used in last equality the fact that X^h be a horizontal process and that every vertical form is the write as $f\omega$, where $f \in C^\infty(P)$ and ω is the connection form. From this and Lemma above we conclude that

$$\int \theta d^v X = \int X^{h*} \theta \delta V - \frac{1}{2} \int \nabla^{G*}(X^{h*}\theta)(dV, dV) = \int X^{h*} \theta d^G V,$$

where the last equality is due to change formula between Stratonovich and Itô integral. Since θ is arbitrary, it follows that X is a vertical martingale if and only if V is a ∇^G -martingale.

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